A Brief Introduction on Simulating Analog Circuit

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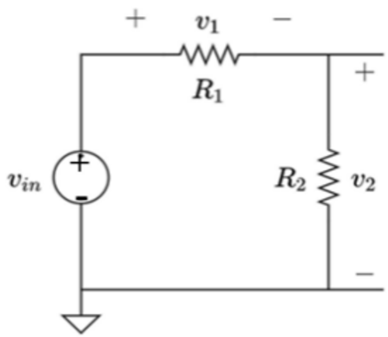
Declaration: Most lessons learnt from Prof. Xuan Zeng and Prof. Ron Rohrer

Figures are mostly borrowed from other sources with appropriate links.

A circuit containing only resistors



Original circuit

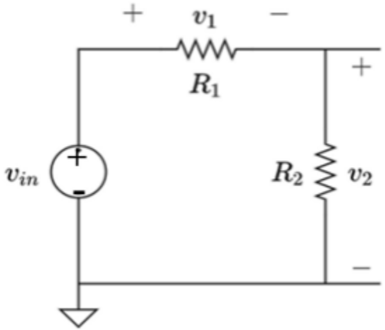
We know the expression of v2: $v_2 = \frac{R_2}{R_1 + R_2} v_{in}$

With this expression, we could directly calculate:

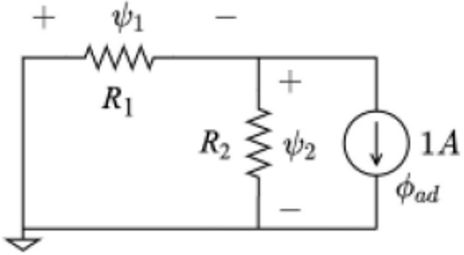
$$\frac{dv_2}{dR_2} = \frac{R_1}{(R_1 + R_2)^2} v_{in}$$

$$\frac{dv_2}{dR_1} = \frac{-R_2}{(R_1 + R_2)^2} v_{in}$$

A circuit containing only resistors



Original circuit {i,v}



Adjoint circuit {$\boldsymbol{\phi}$,$\boldsymbol{\psi}$}

- From Original to Adjoint:
  - short any voltage source
  - open any current source
  - add adjoint excitation source

Based on derivation, adjoint approach states:

$$\delta v_2 = -i_1\phi_1\delta R_1 - i_2\phi_2\delta R_2.$$

Note: use Tellegen's theorem and perturbation, see Jiahua Li et al., *TCAD* 2023.

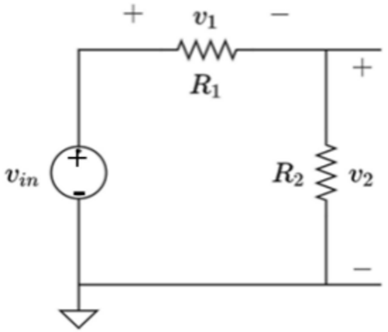
=> Solve the original obtaining {*i*1, *i*2}, and the adjoint obtaining {$\boldsymbol{\phi}$1, $\boldsymbol{\phi}$2}.

=> Then the derivative is straightforward.

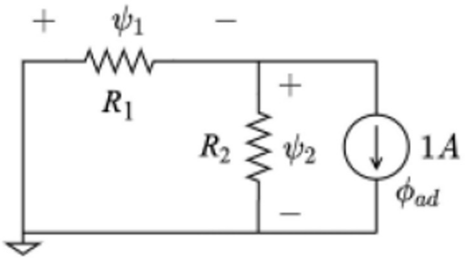
Example:

$$i_1 = \frac{1}{R_1 + R_2} v_{in} \qquad \phi_1 = \frac{R_2}{R_1 + R_2}$$

A circuit containing only resistors

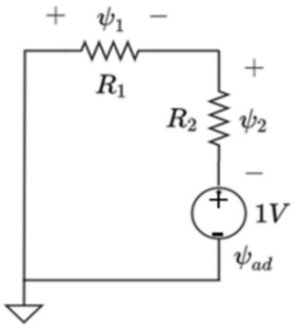


Original circuit {i,v}



Adjoint circuit $\{\boldsymbol{\phi},\boldsymbol{\psi}\}$

For derivative dv2/dR



Adjoint circuit $\{\boldsymbol{\phi},\boldsymbol{\psi}\}$

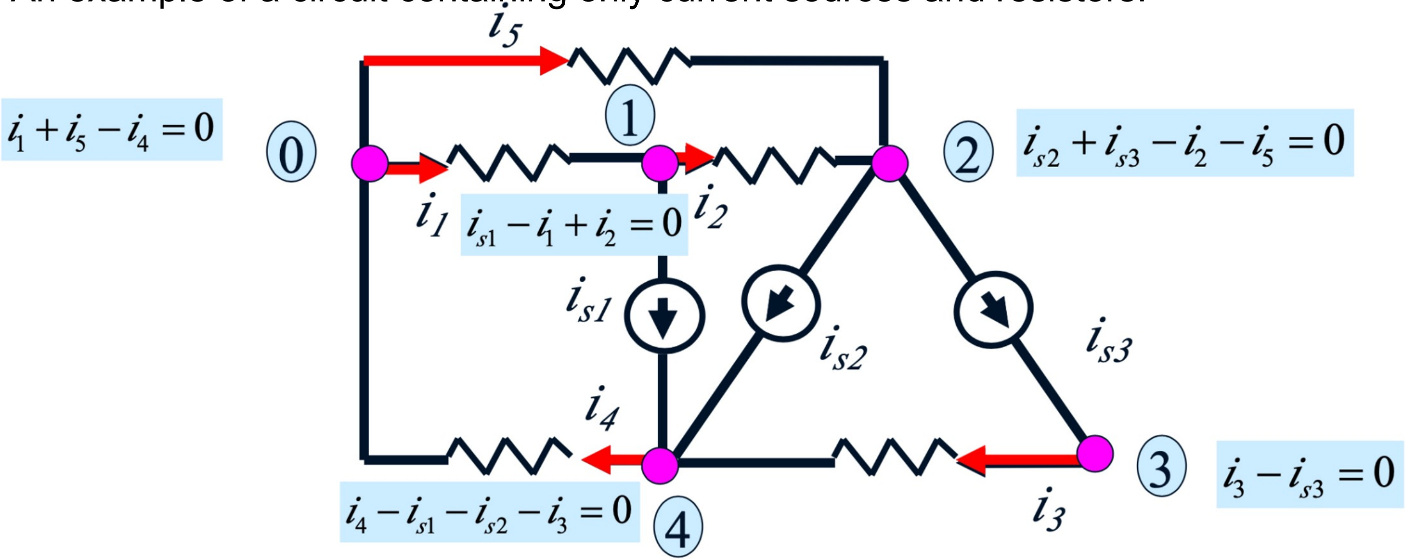
For derivative di2/dR

- ❑ The adjoint excitation source depends on the node of interest and the type of derivative

A circuit containing only resistors

Extending to a large-scale resistive circuit: nodal analysis (NA)

An example of a circuit containing only current sources and resistors:



Kirchoff's current law

Figure credit: Xuan Zeng, Advanced VLSI lecture at Fudan university

A circuit containing only resistors

Extending to a large-scale resistive circuit: nodal analysis

A set of linear equations, could be abstracted as a matrix equation: Gv=i

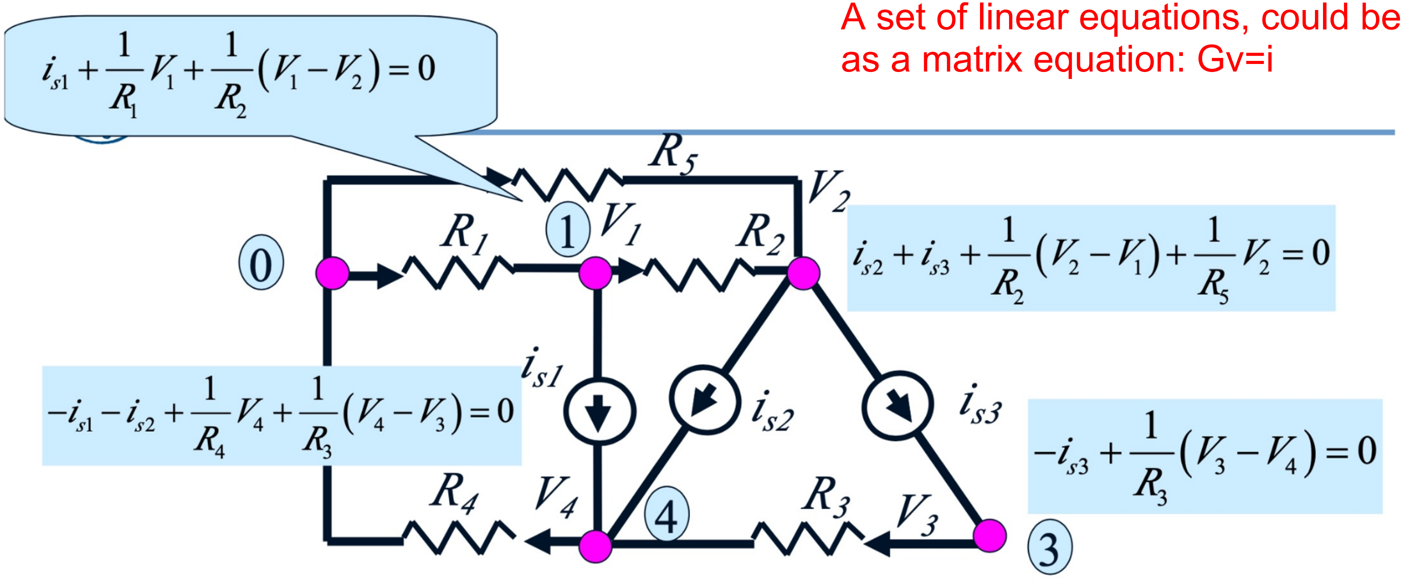


Figure credit: Xuan Zeng, Advanced VLSI lecture at Fudan university

A circuit containing only resistors

Extending to a large-scale resistive circuit: nodal analysis

The circuit matrix G is sparse and symmetric

Stamping: R connected between Node x and Node y => 1/R at (x,x) and (y,y); -1/R at (x,y) and (y,x)

$$
\begin{matrix} \text{Node1} \\ \text{Node2} \\ \text{Node3} \\ \text{Node4} \end{matrix}
\downarrow
\begin{bmatrix}
\frac{1}{R_1}+\frac{1}{R_2} & -\frac{1}{R_2} & & \\
-\frac{1}{R_2} & \frac{1}{R_2}+\frac{1}{R_5} & & \\
 & & \frac{1}{R_3} & -\frac{1}{R_3} \\
 & & -\frac{1}{R_3} & \frac{1}{R_3}+\frac{1}{R_4}
\end{bmatrix}
\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}
=
\begin{bmatrix} -i_{s1} \\ -i_{s2}-i_{s3} \\ i_{s3} \\ i_{s1}+i_{s2} \end{bmatrix}
$$

Figure credit: Xuan Zeng, Advanced VLSI lecture at Fudan university

A circuit containing only resistors

Extending to a large-scale resistive circuit: nodal analysis

The circuit matrix G is sparse and symmetric. Solving Gv=i is done by matrix inverse (core: LU decompostion)

Most importantly, if we are asking dvi/dRj, actually G is same for the original and adjoint. Save run-time!

Node1 Node2 Node3 Node4 (↓)

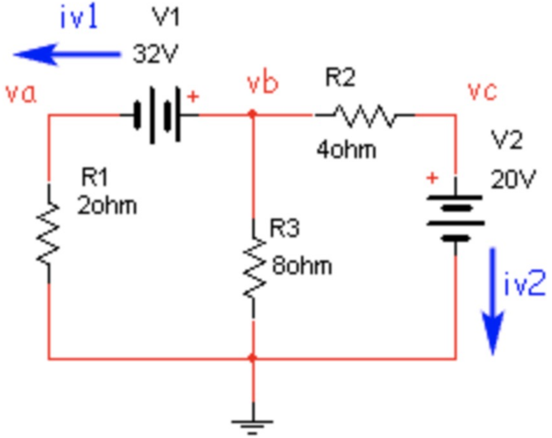
$$\begin{bmatrix} \frac{1}{R_1}+\frac{1}{R_2} & -\frac{1}{R_2} & & \\ -\frac{1}{R_2} & \frac{1}{R_2}+\frac{1}{R_5} & & \\ & & \frac{1}{R_3} & -\frac{1}{R_3} \\ & & -\frac{1}{R_3} & \frac{1}{R_3}+\frac{1}{R_4} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} -i_{s1} \\ -i_{s2}-i_{s3} \\ i_{s3} \\ i_{s1}+i_{s2} \end{bmatrix}$$

Figure credit: Xuan Zeng, Advanced VLSI lecture at Fudan university

A circuit containing only resistors

When a circuit contains voltage source: modified nodal analysis (MNA)

Note: there are alternative approaches, e.g., Norton equivalent (consult Prof. Rohrer)



Two voltage sources, two additional rows/columns

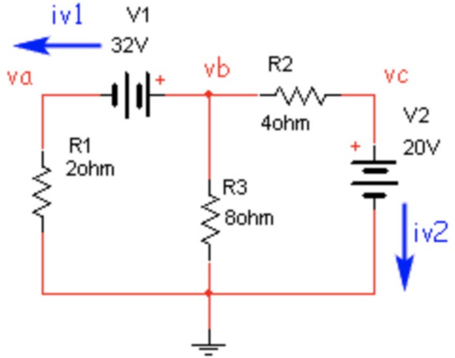
$$\begin{bmatrix} \frac{1}{R_1} & 0 & 0 & -1 & 0 \\ 0 & \frac{1}{R_2}+\frac{1}{R_3} & -\frac{1}{R_2} & 1 & 0 \\ 0 & -\frac{1}{R_2} & \frac{1}{R_2} & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_a \\ v_b \\ v_c \\ i_{v1} \\ i_{v2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ V1 \\ V2 \end{bmatrix}$$

The last row imposes: vc=V2

Figure credit: https://cheever.domains.swarthmore.edu/Ref/mna/MNA2.html

A circuit containing only resistors

Adjoint method review: Solve adjoint circuit once could calculate all dva/dri for any i=1,2,3



$$
\begin{bmatrix}
\frac{1}{R_1} & 0 & 0 & -1 & 0 \\
0 & \frac{1}{R_2}+\frac{1}{R_3} & -\frac{1}{R_2} & 1 & 0 \\
0 & -\frac{1}{R_2} & \frac{1}{R_2} & 0 & 1 \\
-1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{bmatrix}
\begin{bmatrix} v_a \\ v_b \\ v_c \\ i_{v1} \\ i_{v2} \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ V1 \\ V2 \end{bmatrix}
$$

matrix Eq for the original

$$
\begin{bmatrix}
\frac{1}{R_1} & 0 & 0 & -1 & 0 \\
0 & \frac{1}{R_2}+\frac{1}{R_3} & -\frac{1}{R_2} & 1 & 0 \\
0 & -\frac{1}{R_2} & \frac{1}{R_2} & 0 & 1 \\
-1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{bmatrix}
\begin{bmatrix} v_a \\ v_b \\ v_c \\ i_{v1} \\ i_{v2} \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ \boxed{V1} \\ \boxed{V2} \end{bmatrix}
$$

Step1: set to 0

matrix Eq for the adjoint

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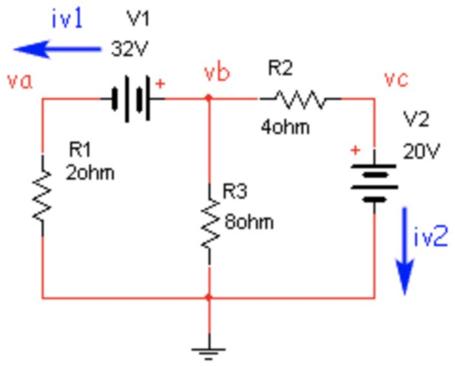
A circuit containing only resistors

Adjoint method review: Solve adjoint circuit once could calculate all dva/dri for any i=1,2,3



$$\begin{bmatrix} \frac{1}{R_1} & 0 & 0 & -1 & 0 \\ 0 & \frac{1}{R_2}+\frac{1}{R_3} & -\frac{1}{R_2} & 1 & 0 \\ 0 & -\frac{1}{R_2} & \frac{1}{R_2} & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_a \\ v_b \\ v_c \\ i_{v1} \\ i_{v2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ V1 \\ V2 \end{bmatrix}$$

matrix Eq for the original

Step2: set to -1

$$\begin{bmatrix} \frac{1}{R_1} & 0 & 0 & -1 & 0 \\ 0 & \frac{1}{R_2}+\frac{1}{R_3} & -\frac{1}{R_2} & 1 & 0 \\ 0 & -\frac{1}{R_2} & \frac{1}{R_2} & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_a \\ v_b \\ v_c \\ i_{v1} \\ i_{v2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ V1 \\ V2 \end{bmatrix}$$

Step1: set to 0

matrix Eq for the adjoint

Remarks: (1) The matrix G on the left doesn't change at all. Reuse the LU decomposition
(2) Solve the adjoint once, then we could get derivative of va w.r.t. any parameters

Figure credit: https://cheever.domains.swarthmore.edu/Ref/mna/MNA2.html

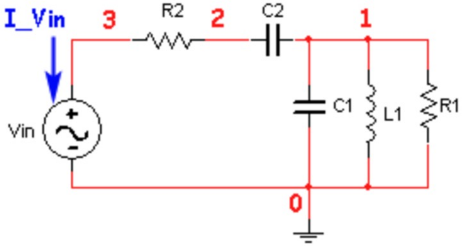
A circuit containing voltage/current sources, resistors, conductors, and inductors

Here we illustrate with Modified nodal analysis (MNA)

It is the same as a pure resistive circuit, if we do MNA in the frequency domain (S domain).

Key: complex impedance. Conductors and inductors are ‘the same as resistors’

$$Z_R = R$$
$$Z_L = sL = j\omega L$$
$$Z_C = \frac{1}{sC} = \frac{1}{j\omega C}$$



$$\begin{bmatrix} s \cdot C1 + s \cdot C2 + \frac{1}{s \cdot L1} + \frac{1}{R1} & -s \cdot C2 & 0 & 0 \\ -s \cdot C2 & s \cdot C2 + \frac{1}{R2} & -\frac{1}{R2} & 0 \\ 0 & -\frac{1}{R2} & \frac{1}{R2} & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} v\_1 \\ v\_2 \\ v\_3 \\ I\_Vin \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ Vin \end{bmatrix}$$

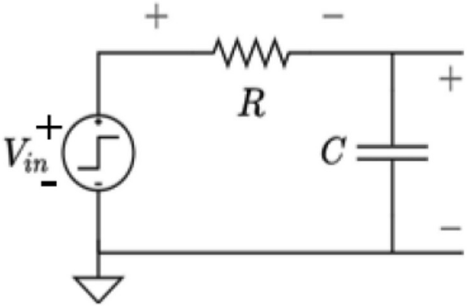
Symbolic solve

Figure credit: https://lpsa.swarthmore.edu/Systems/Electrical/mna/MNA5.html

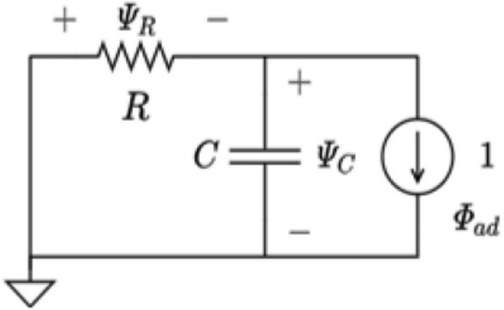
A circuit containing voltage/current sources, resistors, conductors, and inductors

Adjoint method in the frequency domain $\delta V_C = -I_R \Phi_R \delta R + s V_C \Psi_C \delta C.$

Example: See Jiahua Li et al., TCAD 2023.



Original circuit



Adjoint circuit

Note: This adjoint current excitation corresponds to a unit impulse in time domain.

$$I_R(s) = \frac{C}{1 + sRC}$$

$$V_C(s) = \frac{1}{s} \frac{1}{1 + sRC}$$

$$\Phi_R(s) = \frac{1}{1 + sRC}$$

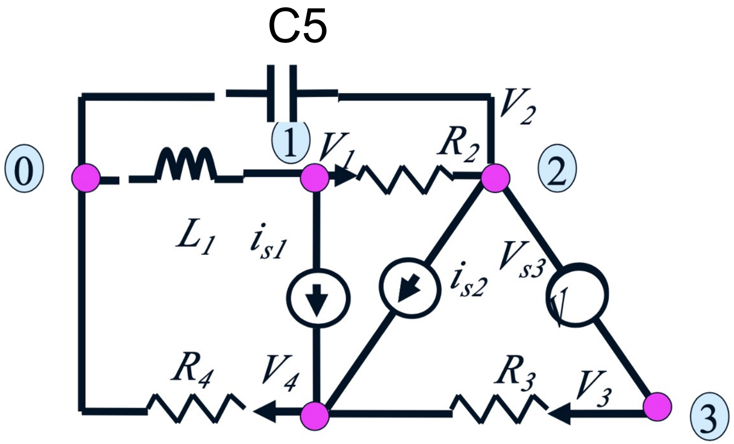
$$\Psi_C(s) = -\frac{R}{1 + sRC}.$$

With the expression in red box, we could get derivatives.

Figure credit: https://lpsa.swarthmore.edu/Systems/Electrical/mna/MNA5.html

A circuit containing voltage/current sources, resistors, conductors, and inductors

Go back to the time domain, an ordinary differential equation (ODE) is needed…



$$
\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & C_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & L_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \dot{V}_3 \\ \dot{V}_4 \\ \dot{i}_{L_1} \\ \dot{i}_{V_{s3}} \end{bmatrix}
+
\begin{bmatrix} \frac{1}{R_2} & -\frac{1}{R_2} & 0 & 0 & -1 & 0 \\ -\frac{1}{R_2} & \frac{1}{R_2} & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{R_3} & -\frac{1}{R_3} & 0 & -1 \\ 0 & 0 & -\frac{1}{R_3} & \frac{1}{R_3}+\frac{1}{R_4} & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \end{bmatrix}
\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ i_{L_1} \\ i_{V_{s3}} \end{bmatrix}
=
\begin{bmatrix} -i_{s1} \\ -i_{s2} \\ 0 \\ i_{s1}+i_{s2} \\ 0 \\ V_{s3} \end{bmatrix}
$$

For simplicity, we denote the equation as: $C\dot{x} + Gx = y$

A linear first-order non-homogeneous ODE system

Figure credit: Xuan Zeng, Advanced VLSI lecture at Fudan university

A circuit containing voltage/current sources, resistors, conductors, and inductors

A linear first-order non-homogeneous ODE system, directly solvable

$$C\dot{x}(t) + Gx(t) = y(t)$$

{C,G,y} are all known, x is unknown

- A general solution: Solve the homogeneous version by calculating eigenvalues and eigenvectors.
- A particular solution: Solve a particular solution to the non-homogeneous system
- Add general and particular solution, determine the unknown coefficients using initial condition of x

Numerical method: Two Integration schemes, (i) explicit: forward Euler (FE), (ii)implicit: backward Euler (BE)

FE $$C\frac{x(t+\Delta t) - x(t)}{\Delta t} = -Gx(t) + y(t)$$

Numerical issue because C is not invertible

BE $$C\frac{x(t+\Delta t) - x(t)}{\Delta t} = -Gx(t+\Delta t) + y(t+\Delta t)$$

$x(t+\Delta t)$ could be solved when $x(t)$ is given

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & C_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & L_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \dot{V}_3 \\ \dot{V}_4 \\ \dot{i}_{L_1} \\ \dot{i}_{V_{s3}} \end{bmatrix} + \begin{bmatrix} \frac{1}{R_2} & -\frac{1}{R_2} & 0 & 0 & 1 & 0 \\ -\frac{1}{R_2} & \frac{1}{R_2} & 0 & 0 & 0 & -1 \\ 0 & 0 & \frac{1}{R_3} & -\frac{1}{R_3} & 0 & 1 \\ 0 & 0 & -\frac{1}{R_3} & \frac{1}{R_3}+\frac{1}{R_4} & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ i_{L_1} \\ i_{V_{s3}} \end{bmatrix} = \begin{bmatrix} -i_{s1} \\ -i_{s2} \\ 0 \\ i_{s1}+i_{s2} \\ 0 \\ V_{s3} \end{bmatrix}$$

C

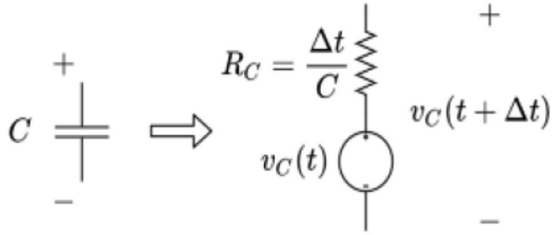
A circuit containing voltage/current sources, resistors, conductors, and inductors

Adjoint method in the time domain

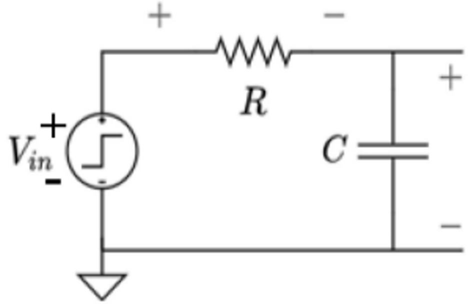
Analytical expression of a capacitor: $i_C(t) = C \cdot \frac{d}{dt} v_C(t).$

BE approximated expression: $i_C(t+\Delta t) \approx C \cdot \frac{v_C(t+\Delta t) - v_C(t)}{\Delta t}$

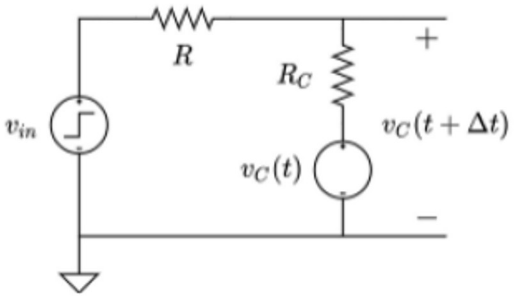
$$v_C(t+\Delta t) \approx \frac{\Delta t}{C} \cdot i_C(t+\Delta t) + v_C(t).$$



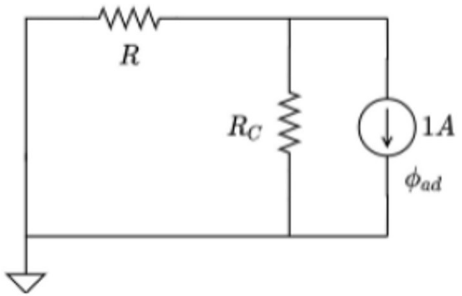
BE companion model for capacitor



original circuit



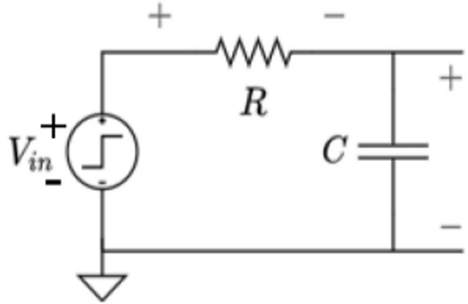
BE companion



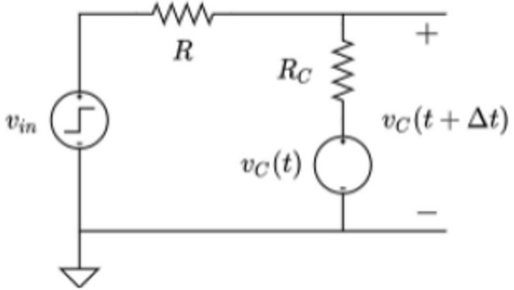
adjoint circuit

A circuit containing voltage/current sources, resistors, conductors, and inductors

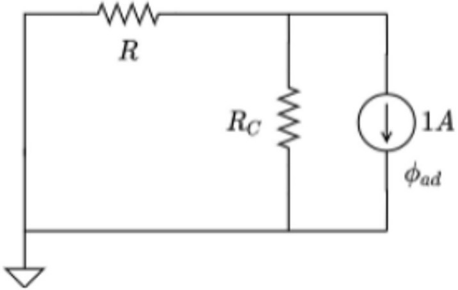
Adjoint method in the time domain



original circuit



BE companion



adjoint circuit

Solve the BE companion circuit: $i_R(t+\Delta t) = i_C(t+\Delta t) = \dfrac{v_{\text{in}}(t+\Delta t) - v_C(t)}{R + \Delta t/C}$

Solve the adjoint circuit:

$$\phi_R(t+\Delta t) = \frac{\Delta t/C}{R+\Delta t/C}$$

$$\phi_C(t+\Delta t) = \frac{-R}{R+\Delta t/C}.$$

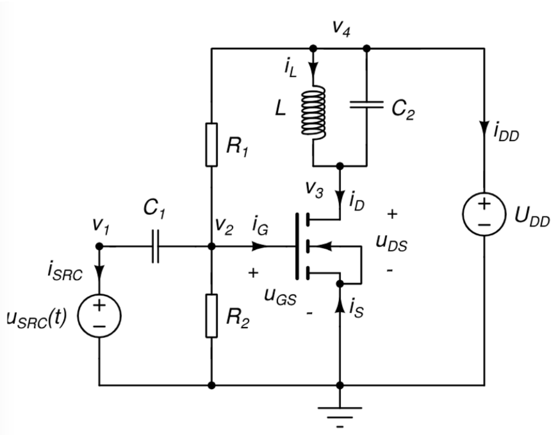
Could yield the exact same results as in Page 13 after some algebra. See Jiahua Li et al., TCAD 2023.

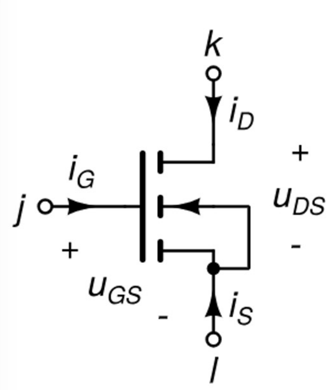
A genernal nonlinear circuit

All previous pages only focus on linear circuits.

Now we turn to a nonlinear circuit (e.g., a circuit contains MOS/BJT)

- ❑ Linearize the nonlinear circuit at each time step with BE (or others)
- ❑ Newton-Raphson method to iteratively find the root





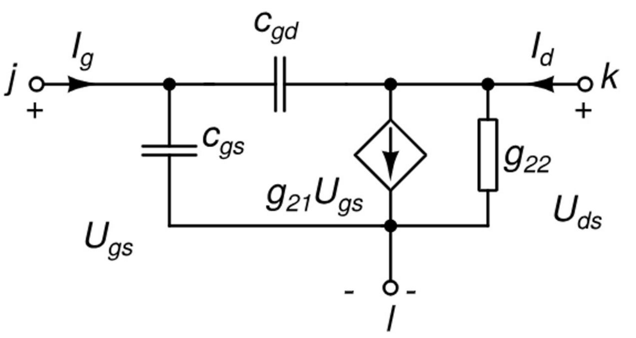


Figure credit: https://fides.fe.uni-lj.si/~arpadb/CAO-old1/

A genernal nonlinear circuit

All previous pages only focus on linear circuits.

Now we turn to a nonlinear circuit (e.g., a circuit contains MOS/BJT)

- ❑ Linearize the nonlinear circuit at each time step with BE (or others)
- ❑ Newton-Raphson method to iteratively find the root (If we believe in linear, then it will be CGS times v2)

multiplication qGS is a function of v2

$$\underbrace{\begin{bmatrix} i_{SRC} \\ R_1^{-1}(v_2 - v_4) + R_2^{-1} v_2 \\ -i_L + K(v_2 - U_T)^2(1 + \lambda v_3) \\ R_1^{-1}(v_4 - v_2) + i_L + i_{DD} \\ v_1 \\ v_4 \\ v_4 - v_3 \end{bmatrix}}_{\mathbf{g}} + \frac{d}{dt} \underbrace{\begin{bmatrix} C_1(v_1 - v_2) \\ C_1(v_2 - v_1) + q_{GD}(v_2 - v_3) + q_{GS}(v_2) \\ -q_{GD}(v_2 - v_3) + C_2(v_3 - v_4) \\ C_2(v_4 - v_3) \\ 0 \\ 0 \\ -L i_L \end{bmatrix}}_{\mathbf{q}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u_{SRC}(t) \\ U_{DD} \\ 0 \end{bmatrix}}_{\mathbf{y}}$$

$$g(x) + \frac{d}{dt} q(x) = y$$

x and y depends on t

A genernal nonlinear circuit

All previous pages only focus on linear circuits.

Now we turn to a nonlinear circuit (e.g., a circuit contains MOS/BJT)

- ❑ Linearize the nonlinear circuit at each time step with BE (or others)
- ❑ Newton-Raphson method to iteratively find the root

$$g(x) + \frac{d}{dt}q(x) = y$$

$$\frac{q(x(t+\Delta t)) - q(x(t))}{\Delta t} = y(t) - g(x(t+\Delta t))$$

A non-linear equation w.r.t. $x(t+\Delta t)$

Many details: truncation error, multi-step, adaptive step, etc,.

A genernal nonlinear circuit

What about adjoint method for a nonlinear circuit?

Still, we first get the BE companion model for the nonlinear circuit (i.e., linearization)

Then we get the adjoint circuit based on the BE companion circuit

Solve the linearized BE companion circuit and the adjoin circuit.

According to the adjoint expression to get the final sensitivity expression.

There are other formulations as well…

The above is trying to obtain sensitivity by simulating both the original and adjoint forward-in-time

We could also simulate the adjoint backward-in-time (i.e., do convolution of the original and the adjoint)

Potential applications of adjoint method in circuit

Most suitable: analog circuit

Actually many analog circuit problems will be beneficial from gradient information

=> yield estimation, yield optimization, fault analysis, circuit optimization

DIRECTOR AND ROHRER: AUTOMATED NETWORK DESIGN 335

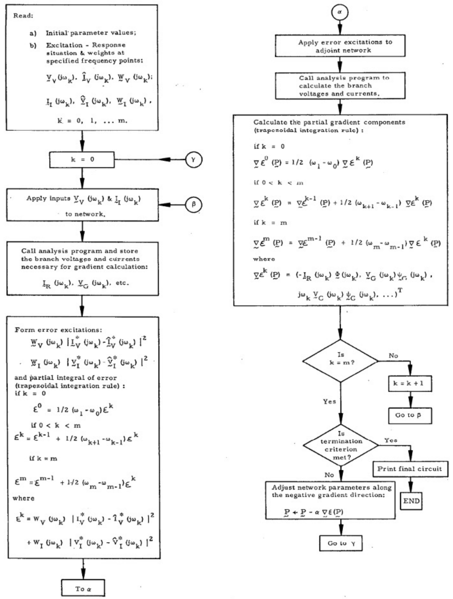


Fig. 3. Flow diagram of frequency-domain design procedure.

As early as 1968 by Diretor and Rohrer

A certain application of adjoint method in ML

Neural Ordinary Differential Equations

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A fully connected layer in neural network: $h_{t+1} = \sigma(W h_t + b)$

Or, equvialently: $h_{t+1} = h_t + f(h_t, \theta)$

Neural ODE: Imagine t could be continuous, the above equation will be described by an ODE!

$$\frac{d\mathbf{h}(t)}{dt} = f(\mathbf{h}(t), t, \theta)$$

If we define a loss L(h(t1)) to be optimized w.r.t. theta, the gradients could be calculated using adjoint method.

References

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